Appendix D

Cumulant Expansions

In this appendix a detailed discussion of the Gersch–Rodriguez cumulant expansion of expectation values of time-ordered integrals is presented. The quantities treated take the form

\[ e^{-E(t)} = \left< \Delta_1 T \exp \left[ i \int_{t_0}^{t} dt' \sum_m \theta_m(t') \right] \Delta_2 \right> \tag{D.1} \]

where both \( \Delta_1, \Delta_2 \) and \( \theta_m \) are arbitrary operators.

It is useful to define here a new operator \( \Gamma_m(t) \) as

\[ 1 - \Gamma_m(t) = T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \tag{D.2} \]

and a new set of operators \( \theta_m(t; \lambda) \) and \( \Gamma_m(t; \lambda) \) that depend on an external parameter \( \lambda \) in terms of which the expansion is to be performed. These satisfy

\[ 1 - \lambda \Gamma_m(t; \lambda) = T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t'; \lambda) \right]. \tag{D.3} \]

The set of operators \( \Gamma_m(t) \) and \( \Gamma_m(t; \lambda) \) can be related to a new collection of operators \( \eta_m(t) \) and \( \eta_m(t; \lambda) \) defined through

\[ \eta_m(t) = 1 - \Gamma_m(t) = T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \tag{D.4} \]

\[ \eta_m(t; \lambda) = 1 - \Gamma_m(t; \lambda) = T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t'; \lambda) \right] \tag{D.5} \]

that obviously satisfy

\[ \frac{\partial \eta(t)}{\partial t} = i \theta_m(t) \eta_m(t) \tag{D.6} \]
\[ \frac{\partial \eta(t; \lambda)}{\partial t} = i\theta_m(t) \eta_m(t; \lambda). \quad \text{(D.7)} \]

Relation (D.4) and (D.5) imply that
\[ \frac{\partial \eta_m(t)}{\partial t} = -\frac{\partial \Gamma_m(t)}{\partial t}, \]
\[ \frac{\partial \eta_m(t; \lambda)}{\partial t} = -\lambda \frac{\partial \Gamma_m(t; \lambda)}{\partial t} = \lambda \frac{\partial \eta_m(t)}{\partial t}, \quad \text{(D.8)} \]

and this means that
\[ \lambda \theta_m(t) \exp \left[ \int_{t_0}^{t} dt' \theta_m(t') \right] = \theta_m(t; \lambda) \exp \left[ \int_{t_0}^{t} dt' \theta_m(t'; \lambda) \right]. \quad \text{(D.9)} \]

This last relation proves to be very useful to derive properties on the \( \lambda \)-dependence of the different operators introduced. The first consequence may be trivially seen by setting \( \lambda = 0 \)
\[ 0 = \theta_m(t; 0) \exp \left[ \int_{t_0}^{t} \theta_m(t'; 0) \right] \quad \text{(D.10)} \]
and this can only be satisfied when
\[ \theta_m(t; \lambda = 0) \equiv \theta_m(t; 0) = 0. \quad \text{(D.11)} \]

On the other hand, setting \( \lambda = 1 \) in eq.(D.3) and comparing to eq.(D.4)
\[ 1 - T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] = 1 - T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t'; 1) \right] \quad \text{(D.12)} \]
one readily verifies that
\[ \theta_m(t) = \theta_m(t; 1). \quad \text{(D.13)} \]

Other quantities required in the expansion are the different \( \lambda \) derivatives of \( \theta_m(t; \lambda) \).
The first order derivative can be computed collecting expressions (D.2) and (D.3) and noticing that
\[ \eta_m(t; \lambda) = 1 - \lambda \Gamma_m(t) = 1 - \lambda (1 - \eta_m(t)), \quad \text{(D.14)} \]
which means that
\[ \eta_m(t; \lambda) = 1 - \lambda + \lambda \eta_m(t). \quad \text{(D.15)} \]
or equivalently
\[ \xi(t; \lambda) \equiv \lambda \theta_m(t) T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \]
\[ = \theta_m(t; \lambda) \left[ 1 - \lambda + \lambda T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t'; \lambda) \right] \right]. \quad \text{(D.16)} \]
The first $\lambda$ derivative of $\xi(t;\lambda)$ reads

$$
\frac{\partial \xi(t;\lambda)}{\partial \lambda} = \theta_m(t;\lambda) T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] 
$$

and evaluated at $\lambda = 0$

$$
\left. \frac{\partial \xi(t;\lambda)}{\partial \lambda} \right|_{\lambda=0} = \theta_m(t) T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] 
$$

as it has already been seen in eq. (D.11) that $\theta_m(t;0) = 0$. Hence

$$
\left. \frac{\partial \theta_m(t;\lambda)}{\partial \lambda} \right|_{\lambda=0} = \theta_m(t) T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] .
$$

The second $\lambda$ derivative of $\theta_m(t;\lambda)$ can be computed proceeding in the same way. Starting from expression (D.16) one easily verifies that

$$
\left. \frac{\partial^2 \xi(t;\lambda)}{\partial \lambda^2} \right|_{\lambda=0} = \frac{\partial}{\partial \lambda} \left( \frac{\partial \xi(t;\lambda)}{\partial \lambda} \right) |_{\lambda=0} = \frac{\partial}{\partial \lambda} \left[ \theta_m(t) T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \right] \equiv 0
$$

and evaluated at $\lambda = 0$

$$
\left. \frac{\partial^2 \theta_m(t;\lambda)}{\partial \lambda^2} \right|_{\lambda=0} = \frac{\partial^2 \theta_m(t;\lambda)}{\partial \lambda^2} \left[ 1 - \lambda + \lambda T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \right] + 2 \frac{\partial \theta_m(t;\lambda)}{\partial \lambda} \left[ -1 + T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \right] .
$$

which yields, once evaluated at $\lambda = 0$

$$
\left. \frac{\partial^2 \xi(t;\lambda)}{\partial \lambda^2} \right|_{\lambda=0} = \left. \frac{\partial^2 \theta_m(t;\lambda)}{\partial \lambda^2} \right|_{\lambda=0} + 2 \left. \frac{\partial \theta_m(t;\lambda)}{\partial \lambda} \right|_{\lambda=0} [\eta_m(t) - 1] = 0 .
$$

Substituting $\eta_m(t)$ by its explicit form given in eq. (D.3), one finally arrives at

$$
\left. \frac{\partial^2 \theta_m(t;\lambda)}{\partial \lambda^2} \right|_{\lambda=0} = 2 \theta_m(t) T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \left[ 1 - T \exp \left[ i \int_{t_0}^{t} dt' \theta_m(t') \right] \right] .
$$
Higher order derivatives of $\theta_m(t; \lambda)$ could be obtained following the same steps. The expansion of eq. (D.1) can be derived introducing a new function $E(t; \lambda)$ such that

$$e^{-E(t; \lambda)} = \left\langle \Delta_1 T \exp \left[ i \int_0^t dt' \sum_m \theta_m(t'; \lambda) \right] \Delta_2 \right\rangle$$  \hspace{1cm} (D.23)

where

$$E(t; \lambda) = - \ln \left[ \left\langle \Delta_1 T \exp \left[ i \int_0^t dt' \sum_m \theta_m(t'; \lambda) \right] \Delta_2 \right\rangle \right] .$$  \hspace{1cm} (D.24)

Due to the behavior of $\theta_m(t; \lambda)$ at $\lambda=0$, $E(t; \lambda = 0)$ becomes

$$E(t; 0) = - \ln \left\langle \Delta_1 \Delta_2 \right\rangle ,$$  \hspace{1cm} (D.25)

while due to the behavior of $\theta_m(t; \lambda)$ at $\lambda=1$

$$E(t; 1) = - \ln \left[ \left\langle \Delta_1 T \exp \left[ i \int_0^t dt' \sum_m \theta_m(t') \right] \Delta_2 \right\rangle \right]$$  \hspace{1cm} (D.26)

and this is precisely the definition of $E(t)$.

These two last equations indicate that $E(t)$ can be recovered from the $\lambda$ expansion of $E(t; \lambda)$ around $\lambda = 0$. However, $E(t; \lambda)$ may not be well behaved and therefore the expansion may be singular. One thus has to assume that this will not be the case, and that $E(t; \lambda)$ is analytic in the unit circle. Whether this is true or not depends on the explicit analytic form of the ground state wave function and the operators entering in eq. (D.1).

The working hypothesis is, therefore, that $E(t; \lambda)$ can safely expanded in a McLaurin series

$$E(t; \lambda) = E(t; 0) + \frac{\partial E(t; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} + \frac{1}{2!} \frac{\partial^2 E(t; \lambda)}{\partial \lambda^2} \bigg|_{\lambda=0} + \ldots$$  \hspace{1cm} (D.27)

where the first coefficient reads

$$E(t; 0) = - \ln \left\langle \Delta_1 \Delta_2 \right\rangle .$$  \hspace{1cm} (D.28)

The derivatives of $E(t; \lambda)$ can be obtained from the derivatives of $\theta_m(t; \lambda)$. As a matter of fact, denoting

$$\Omega(t; \lambda) = T \exp \left[ i \int_0^t dt' \sum_m \theta_m(t'; \lambda) \right]$$  \hspace{1cm} (D.29)

one readily notices that

$$\frac{\partial \Omega(t; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} = \frac{\partial}{\partial \lambda} \left[ 1 + i \int_0^t dt' \sum_m \theta_m(t'; \lambda) \right]$$
\[ + i^2 \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_1} dt_2 \sum_m \sum_n \theta_m(t_1; \lambda) \theta_n(t_2; \lambda) + \ldots \bigg|_{\lambda=0} = i \sum_m \int_{t_0}^{t_1} dt' \frac{\partial \theta_m(t'; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} + i^2 \sum_n \sum_n \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_1} dt_2 \frac{\partial}{\partial \lambda} \left( \theta_m(t_1; \lambda) \theta_n(t_2; \lambda) \right) \bigg|_{\lambda=0} + \ldots \] (D.30)

This expression can be simplified recalling the result of eq.(D.19). As \( \theta_m(t; 0) = 0 \), all \( n \)-order derivatives of products of \( kn \theta_m(t; \lambda) \) operators vanish. In particular

\[ \frac{\partial}{\partial \lambda} \left[ \theta_0(t; \lambda) \theta_\beta(t; \lambda) \right] \bigg|_{\lambda=0} = \frac{\partial \theta_\alpha(t; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} + \theta_\alpha(t; 0) \frac{\partial \theta_\beta(t; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} = 0 . \] (D.31)

As a consequence, only the first term of eq.(D.30) contributes. The first derivative of \( \Omega(t; \lambda) \) then reads

\[ \frac{\partial \Omega(t; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} = i \sum_m \int_{t_0}^{t_1} dt' \theta_m(t')T \exp \left[ i \int_{t_0}^{t_1} dt'' \theta_m(t'') \right] = \sum_m \left[ i \int_{t_0}^{t_1} dt' \theta_m(t') + i^2 \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_1} dt_2 \theta_m(t_1) \theta_m(t_2) \right] + \ldots \]

\[ = \sum_m \left[ T \exp \left[ i \int_{t_0}^{t_1} dt' \theta_m(t') \right] - 1 \right] \equiv - \sum_m \Gamma_m(t) . \] (D.32)

The second \( \lambda \) derivative of \( \Omega(t; \lambda) \) can be computed following the same steps. In fact, it is easy to show that

\[ \frac{\partial^2 \Omega(t; \lambda)}{\partial \lambda^2} \bigg|_{\lambda=0} = i \sum_m \int_{t_0}^{t_1} dt' \frac{\partial^2 \theta_m(t'; \lambda)}{\partial \lambda^2} \bigg|_{\lambda=0} + 2i^2 \sum_n \sum_n \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_1} dt_2 \frac{\partial \theta_m(t_1; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} \frac{\partial \theta_n(t_2; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} , \] (D.33)

which, together with eqs.(D.18) and (D.22), becomes

\[ \frac{\partial^2 \Omega(t; \lambda)}{\partial \lambda^2} \bigg|_{\lambda=0} = 2i \sum_m \int_{t_0}^{t_1} dt_1 \theta_m(t_1) T \exp \left[ i \int_{t_0}^{t_1} dt_2 \theta_m(t_2) \right] \times \left[ 1 - T \exp \left[ i \int_{t_0}^{t_1} dt_3 \theta_m(t_3) \right] \right] \]

\[ + 2i^2 \sum_n \sum_n \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_1} dt_2 \sum_m \theta_m(t_1) T \exp \left[ i \int_{t_0}^{t_1} dt_3 \theta_m(t_3) \right] \times \sum_n \theta_n(t_2) T \exp \left[ i \int_{t_0}^{t_2} dt_4 \theta_m(t_4) \right] . \] (D.34)
In terms of the $\Gamma_m(t)$ functions defined in eq. (D.2), the last equation reads
\[
\left. \frac{\partial^2 \Omega(t; \lambda)}{\partial \lambda^2} \right|_{\lambda = 0} = 2 \sum_m \int_{t_0}^t dt' \left. \frac{\partial \Gamma_m(t')}{\partial t'} \right|_{t = t'} \Gamma_m(t') + 2 \sum_{m} \sum_{n} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \left. \frac{\partial \Gamma_m(t_1)}{\partial t_1} \right|_{t_1 = t_1} \left. \frac{\partial \Gamma_n(t_2)}{\partial t_2} \right|_{t_2 = t_2} . \tag{D.35}
\]

Once with the lowest order derivatives of $\Gamma(t; \lambda)$, the first terms in the McLaurin
expansion of $E(t; \lambda)$ can be written down. Recalling that
\[
E(t; \lambda) = -\ln \left< \Delta_1 \Omega(t; \lambda) \Delta_2 \right> , \tag{D.36}
\]
one has
\[
\left. \frac{\partial E(t; \lambda)}{\partial \lambda} \right|_{\lambda = 0} = -\frac{1}{\left< \Delta_1 \Omega(t; \lambda) \Delta_2 \right>} \left< \Delta_1 \left. \frac{\partial \Omega(t; \lambda)}{\partial \lambda} \right|_{\lambda = 0} \Delta_2 \right> \tag{D.37}
\]
and therefore
\[
\left. \frac{\partial E(t; \lambda)}{\partial \lambda} \right|_{\lambda = 0} = \sum_m \frac{\left< \Delta_1 \Gamma_m(t) \Delta_2 \right>}{\left< \Delta_1 \Delta_2 \right>} . \tag{D.38}
\]

The second derivative of $E(t; \lambda)$ can be also expressed in terms of the $\Gamma_m(t)$ functions
as follows
\[
\left. \frac{\partial^2 E(t; \lambda)}{\partial \lambda^2} \right|_{\lambda = 0} = 2 \left< \Delta_1 \Delta_2 \right> \left[ \left. \frac{\partial}{\partial \lambda} \left< \Delta_1 \Omega(t; \lambda) \Delta_2 \right> \right|_{\lambda = 0} \right]^2 - \frac{1}{\left< \Delta_1 \Delta_2 \right>} \left[ \left. \frac{\partial^2}{\partial \lambda^2} \left< \Delta_1 \Omega(t; \lambda) \Delta_2 \right> \right|_{\lambda = 0} \right] \left< \Delta_1 \Delta_2 \right> \nonumber \]
\[
= \frac{1}{\left< \Delta_1 \Delta_2 \right>} \left[ \sum_m \left< \Delta_1 \Gamma_m(t) \Delta_2 \right> \right]^2 - \frac{1}{\left< \Delta_1 \Delta_2 \right>} \left[ \sum_m \left< \Delta_1 \Gamma_m(t) \Delta_2 \right> \right]^2 - \frac{1}{\left< \Delta_1 \Delta_2 \right>} \sum_m \left< \Delta_1 \int_{t_0}^t dt' \left. \frac{\partial \Gamma_m(t')}{\partial t'} \right|_{t' = t'} \Gamma_m(t') \Delta_2 \right> \nonumber \]
\[
- \sum_{m} \sum_{n} \left< \Delta_1 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \left. \frac{\partial \Gamma_m(t_1)}{\partial t_1} \right|_{t_1 = t_1} \left. \frac{\partial \Gamma_n(t_2)}{\partial t_2} \right|_{t_2 = t_2} \Delta_2 \right> . \tag{D.39}
\]

The integral on the fourth line of eq. (D.39) can be easily carried out, yielding a
factor $\Gamma_m^2(t)$ which exactly cancel with the $m \neq n$ terms of the third term. As a result
one finally gets
\[
\left. \frac{\partial^2 T(t; \lambda)}{\partial \lambda^2} \right|_{\lambda = 0} = \frac{1}{\left< \Delta_1 \Delta_2 \right>} \left[ \sum_m \left< \Delta_1 \Delta_2 \right> \right]^2 - \frac{1}{\left< \Delta_1 \Delta_2 \right>} \sum_{m \neq n} \left< \Delta_1 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \left. \frac{\partial \Gamma_m(t_1)}{\partial t_1} \right|_{t_1 = t_1} \left. \frac{\partial \Gamma_n(t_2)}{\partial t_2} \right|_{t_2 = t_2} \Delta_2 \right> . \tag{D.40}
\]
Cumulant Expansions

Equations (D.28), (D.39) and (D.40) show the first coefficients of the series expansion of \( E(t; \lambda) \). Higher order terms can be computed in the same way, even though the derivations gets more involved at each step. In general, however, the coefficients \( \omega_k \) of the series expansion of \( E(t; \lambda) \)

\[
\omega_k = -\frac{1}{k!} \frac{\partial E(t; \lambda)}{\partial \lambda} \bigg|_{\lambda=0}, \quad \forall k \geq 1 \tag{D.41}
\]

can be used to write

\[
E(t; \lambda) = -\ln \langle \Delta_1 \Delta_2 \rangle - \sum_{k=1}^{\infty} \omega_k \tag{D.42}
\]

and the cumulant expansion of \( e^{-E(t; \lambda)} \) becomes

\[
e^{-E(t; \lambda)} = \langle \Delta_1 \Delta_2 \rangle \exp \left[ \lambda \omega_1 + \lambda^2 \omega_2 + \lambda^3 \omega_3 + \ldots \right]. \tag{D.43}
\]

Evaluated at \( \lambda = 1 \), this last expression yields the desired expansion

\[
\left\langle \Delta_1 T \exp \left[ i \int_{t_0}^t dt' \sum_m \theta_m(t') \right] \Delta_2 \right\rangle \equiv \langle \Delta_1 \Delta_2 \rangle 
\times \exp \left[ -\sum_m \frac{\langle \Delta_1 \left( 1 - T \exp \left[ \int_{t_0}^t dt' \theta_m(t') \right] \right) \Delta_2 \rangle}{\langle \Delta_1 \Delta_2 \rangle} + \ldots \right]. \tag{D.44}
\]

Consider the particular case in which all \( \theta_m(t) \) operators commute among themselves and with \( \Delta_2 \), in such a way that the whole expression in the l.h.s. of eq. (D.1) admits an expression of the form

\[
\int dr^N f(r_1, r_2, \ldots, r_N; r'_1) \exp \left[ i \sum_m \int_{t_0}^t dy \theta_m(y) \right] \tag{D.45}
\]

where the function \( f \) is the result of the simultaneous action of \( \Delta_1 \Delta_2 \) on the groundstate wave function, and \( x, y \) and \( y \) are some combination of \( t \) and \( t' \). Then, the Gersh–Rodriguez cumulant expansion of a quantity of the form

\[
\phi(r_1, r'_1) + \int dr^N f(r_1, r_2, \ldots, r_N; r'_1) \exp \left[ i \sum_m \int_{t_0}^t dy \theta_m(y) \right] \tag{D.46}
\]

reads

\[
W_0 \exp [\omega_1 + \omega_2 + \omega_3 + \ldots], \tag{D.47}
\]

where

\[
W_0 = \phi(r_1, r'_1) + \int dr^N f(r_1, r_2, \ldots, r_N; r'_1) \tag{D.48}
\]

\[
\omega_1 = -\frac{1}{W_0} \sum_m \int dr^N f(r_1, r_2, \ldots, r_N; r'_1) \left[ 1 - \exp \left( i \int_{t_0}^x dy \theta_m(y) \right) \right] \tag{D.49}
\]

\[
\omega_2 = \ldots \tag{D.50}
\]

are the first cumulants.