Appendix C

Proof of the Identity (4.12)

The following identity is proved in this appendix

\[ T \exp \left[ iH(t - t_0) + i \int_{t_0}^{t} dt' U_j(v_q t') \right] \]

\[ = T \exp \left[ i \int_{t_0}^{t} dt' e^{iH_0(t - t')} U_j(v_q t') e^{-iH_0(t - t')} \right] e^{iH_0(t - t_0)} \]  \hspace{1cm} (C.1)

The proof can be derived defining a new operator \( \Lambda(t - t_0) \) that equals the time-ordered integral of the l.h.s. of (C.1)

\[ \Lambda(t) = T \exp \left[ iH(t - t_0) + i \int_{t_0}^{t} dt' U_j(v_q t') \right] , \]  \hspace{1cm} (C.2)

and showing that there is a first order differential equation satisfied by \( \Lambda(t) \) whose solution can be written as in the r.h.s. of (C.1). The time derivative of \( \Lambda(t) \) reads

\[ \frac{d \Lambda(t)}{dt} = \frac{d}{dt} T \exp \left[ iH(t - t_0) + i \int_{t_0}^{t} dt' U_j(v_q t') \right] \]

\[ = \frac{d}{dt} T \exp \left[ i \int_{t_0}^{t} dt' \left[ H + U_j(v_q t') \right] \right] \]

\[ = i \left[ H + U_j(v_q t) \right] T \exp \left[ i \int_{t_0}^{t} dt' \left[ H + U_j(v_q t') \right] \right] , \]  \hspace{1cm} (C.3)

and therefore

\[ \frac{d \Lambda(t)}{dt} = i \left[ H + U_j(v_q t) \right] \Lambda(t) \]  \hspace{1cm} (C.4)
which must be solved together with the initial condition at $t = t_0$

$$\Lambda(t_0) = 1$$  \hspace{1cm} (C.5)

obtained from (C.2).

Alternative representation of $\Lambda(t)$ results from the alternative representations that the solution of eq. (C.6) can take. Consider the following expression proposed for $\Lambda(t)$

$$\Lambda(t) = e^{iH(t-t_0)}O(t)$$  \hspace{1cm} (C.6)

which defines in fact the new operator $O(t)$. The first derivative of $\Lambda(t)$ then reads

$$\frac{d\Lambda(t)}{dt} = iHe^{iH(t-t_0)}O(t) + e^{iH(t-t_0)}\frac{dO(t)}{dt}. \hspace{1cm} (C.7)$$

Introducing this in eq.(C.4) the differential equation for $\Lambda(t)$ becomes a differential equation for $O(t)$

$$\frac{dO(t)}{dt} = ie^{-iH(t-t_0)}U_j(v_q t)e^{iH(t-t_0)}O(t). \hspace{1cm} (C.8)$$

which must be solved with the initial condition $O(t_0) = 1$ obtained from eq. (C.5) and (C.6). A formal solution to eq.(C.8) may be given in the form of a Dyson series

$$O(t) = 1 + i \int_{t_0}^{t} dt' e^{-iH(t'-t_0)}U_j(v_q t')e^{iH(t'-t)}O(t') \hspace{1cm} (C.9)$$

which implies that

$$\Lambda(t) = e^{iH(t-t_0)} + i \int_{t_0}^{t} dt' e^{iH(t-t')U_j(v_q t')e^{iH(t'-t)}e^{iH(t-t_0)}} \hspace{1cm} (C.10)$$

Now, the successive iteration of eq. (C.10) results in the expression

$$\Lambda(t) = e^{iH(t-t_0)} + i \int_{t_0}^{t} dt' e^{iH(t-t')U_j(v_q t')e^{iH(t'-t)}e^{iH(t-t_0)}} +$$

$$+ i^2 \int_{t_0}^{t} dt'_1 \int_{t_0}^{t_1} dt'_2 e^{iH(t-t_1)U_j(v_q t_1) e^{iH(t_1-t_2)}U_j(v_q t_2) e^{iH(t-t_2)}e^{iH(t-t_0)}} + \ldots =$$

$$\left[ 1 + \int_{t_0}^{t} dt' e^{iH(t-t')U_j(v_q t')e^{-iH(t-t')}} +$$

$$+ i^2 \int_{t_0}^{t} dt'_1 \int_{t_0}^{t_1} dt'_2 e^{iH(t-t_1)U_j(v_q t_1) e^{-iH(t_1-t_2)}U_j(v_q t_2) e^{-iH(t-t_2)}e^{-iH(t-t_0)}} + \ldots \right] e^{iH(t-t_0)}$$

$$\equiv T \exp \left[ i \int_{t_0}^{t} dt' e^{iH(t-t')U_j(v_q t')e^{-iH(t-t')}} \right] e^{iH(t-t_0)} \hspace{1cm} (C.11)$$

which proves that $\Lambda(t)$ may also be written as indicated in the r.h.s. of (C.1).